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FINITE DEFLECTION DYNAMICS OF ELASTIC BEAMS

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Abstract—Solutions are obtained for the problem of an infinite elastic beam subjected to essentially constant velocity boundary conditions at one point of the beam. The effects of finite deflections, normal force, rotatory inertia and shear deformation are included. The equations of the problem are converted into non-dimensional form and a perturbation approach is used to obtain a consistent approximation. Numerical solutions are obtained for the bending moment, shear force and the normal force for different velocities of impact. It is shown that the solution to the problem depends on a combined geometrical and material parameter which does not vary significantly for compact sections and a loading parameter which determines the amplitude of the response. Finally the linear Timoshenko beam theory is shown to predict the bending moment and shear force extremely well even when the deflections are large enough to cause appreciable stretching of the centroidal axis.

INTRODUCTION

Timoshenko beam theory has been widely used in the analysis of the dynamics of elastic beams. Unlike the elementary Euler–Bernoulli analysis, the Timoshenko beam theory includes the effects of shear deflections and rotatory inertia and leads to hyperbolic partial differential equations. Transform techniques have been used to derive solutions for semi-infinite Timoshenko beams for a variety of end loadings by Miklowitz [1], and Boley and Chao [2]. Plass [3], and Leonard and Budiansky [4] employed characteristic theory to solve the Timoshenko beam equations for pulse type loading. Results from the Timoshenko beam theory have been shown to be in good agreement with experimental observations by Goland, Wickersham and Dengler [5], and more recently by Ranganath [6]. However, these studies do not include the non-linear effects of large deflection and the membrane force resulting from the stretching of the middle surface. The purpose of the present paper is to provide a more general analysis which includes these effects in order to determine the range of validity of the linear small deflection theory. Particular attention is given to the question (see e.g. Durelli [7]) of the importance of membrane stresses in the response of a beam subjected to transverse impact.

Lee [8] has derived a system of first order partial differential equations governing the dynamics of beams subjected to finite deflections but sufficiently small strains for linear elastic material behavior to be an appropriate idealization. His analysis includes the effects of large deflections, membrane force, shear deformation and rotatory inertia. He has shown that the system of equation is hyperbolic; that two of the three non-zero characteristic speeds are equal, and equal to the elastic bar velocity; that for small values of the membrane tension the third characteristic speed reduces to the velocity of shear waves in Timoshenko beam theory whereas for large values of membrane tension it reduces to the velocity of waves in an inextensible string.

In the present analysis, equations analogous to those derived by Lee [8] have been solved approximately by using a perturbation theory approach to simplify the equations and integrating the resulting equations by means of finite differences. Numerical solutions are obtained for the case in which the transverse velocity of a point on an infinite beam is increased rapidly to a prescribed value and then maintained constant thereafter.

Minor differences exist between equations (8) herein and corresponding dimensionless counterparts of Lee's equations [8]. These differences are due to different choices with regard to three aspects of the two formulations. First, in the present paper the authors have chosen to regard stress resultants as three of the dependent variables instead of the three deformation measures empolyed by Lee. Second, the authors have taken the constitutive equations to be linear relationships between material time derivatives of stress resultants and rates of deformation relative to the current configuration, whereas Lee assumed a linear relationship between stress resultants and the corresponding generalized strain. Third, the authors have used the arc length s along the beam in the current configuration as the independent space variable whereas Lee used distance S in the undeformed configuration. The two formulations are equivalent in all important respects for the small strain case assumed in both investigations. Stress resultants have been chosen as three of the dependent variables in the present investigation because these quantities appear directly in boundary conditions where forces and moments are prescribed. The assumption of a constitutive equation which is linear in stress resultant rates and deformation rates as opposed to linear in the stress resultants and deformation measures themselves is made because it is expected that the former assumption is a good approximation for larger extensions than the latter. This expectation is based on the fact that the former assumption exhibits a reduction in stiffness of a tensile bar with increasing extension, which is to be expected due to reduction of the cross-sectional area through lateral contraction. The choice of the current length s as the space coordinate is made because the rate of deformation measures used in the constitutive equations are expressed more conveniently in terms of derivatives with respect to s than with respect to the initial length S.

GOVERNING EQUATIONS AND STATEMENT OF PROBLEM

Consider the plane deformation of a beam loaded in one of its planes of symmetry. Let the velocity at time t of a point on the centroidal axis of the beam at s (see Fig. 1a) be denoted by the velocity vector $\mathbf{V}(x, t)$ given by

$$\mathbf{V} = v\boldsymbol{\eta} + u\boldsymbol{\xi} \tag{1}$$

where $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ are unit vectors normal and tangential, respectively, to the centroidal axis at s; v and u are therefore the normal and tangential components of the velocity of a point on the centroidal axis. Taking the material time derivative† of (1), and making use of the

† The material time derivative $\dot{f}$ of a function $f(s, t)$ is defined by

$$\dot{f} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial s}.$$

relations

$$\dot{\boldsymbol{\eta}} = -\left(\frac{\partial v}{\partial s} + u\frac{\partial \theta}{\partial s}\right)\boldsymbol{\xi} \tag{2a}$$

$$\dot{\boldsymbol{\xi}} = \left(\frac{\partial v}{\partial s} + u\frac{\partial \theta}{\partial s}\right)\boldsymbol{\eta} \tag{2b}$$

where $\partial\theta/\partial s$ is the curvature of the centroidal axis, we obtain the following expression for the acceleration vector

$$\dot{\mathbf{V}} = \left\{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial s} + u\left(\frac{\partial v}{\partial s} + u\frac{\partial \theta}{\partial s}\right)\right\}\boldsymbol{\eta} + \left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s} - v\left(\frac{\partial v}{\partial s} + u\frac{\partial \theta}{\partial s}\right)\right\}\boldsymbol{\xi}. \tag{3}$$

As in the linear Timoshenko beam theory [9] a cross-section of the beam is assumed to remain plane, but not necessarily perpendicular to the centroidal axis. The angular velocity of a cross-section is denoted by $\omega(s, t)$.

Let the force acting on a cross-section perpendicular to the centroidal axis (see Fig. 1b) be denoted by the vector $\mathbf{F}(s, t)$ given by

$$\mathbf{F} = Q\boldsymbol{\eta} + N\boldsymbol{\xi} \tag{4}$$

in which Q is the shear force in the direction $\boldsymbol{\eta}$ and N is the normal force (membrane force) in the direction $\boldsymbol{\xi}$. Let the bending moment about an axis through the centroid be denoted by $M(s, t)$. Then, in terms of the generalized particle velocities u, v, ω and the generalized stresses N, Q, M defined previously, the following equations of motion can be derived by consideration of the beam element shown in Fig. 1(b).

$$\frac{\partial M}{\partial s} + Q = \rho I\left\{\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial s}\right\} \tag{5a}$$

$$\frac{\partial Q}{\partial s} + N\frac{\partial \theta}{\partial s} = \rho A\left\{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial s} + u\left(\frac{\partial v}{\partial s} + u\frac{\partial \theta}{\partial s}\right)\right\} \tag{5b}$$

$$\frac{\partial N}{\partial s} - Q\frac{\partial \theta}{\partial s} = \rho A\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s} - v\left(\frac{\partial v}{\partial s} + u\frac{\partial \theta}{\partial s}\right)\right\} \tag{5c}$$

In equations (5), ρ is mass density, A is cross-sectional area and I is the moment of inertia of the beam cross-section about the centroidal axis perpendicular to $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$.

In addition to the equations of motion (5) the governing system of equations consists of one kinematical condition and three constitutive equations. The kinematical condition is

$$\frac{\partial \theta}{\partial t} = \frac{\partial v}{\partial s} \tag{6}$$

which follows directly from (2) and either of the identities $\dot{\boldsymbol{\xi}} = \dot{\theta}\boldsymbol{\eta}$ or $\dot{\boldsymbol{\eta}} = -\dot{\theta}\boldsymbol{\xi}$. The three constitutive equations are taken to be the three linear rate equations

$$\dot{M} = EI\frac{\partial \omega}{\partial s} \tag{7a}$$

$$\dot{Q} = \kappa(\dot{\theta} - \omega) = \kappa\left(\frac{\partial v}{\partial s} + u\frac{\partial \theta}{\partial s} - \omega\right) \tag{7b}$$

$$\dot{N} = EA\left(\frac{\partial u}{\partial s} - v\frac{\partial \theta}{\partial s}\right) \tag{7c}$$

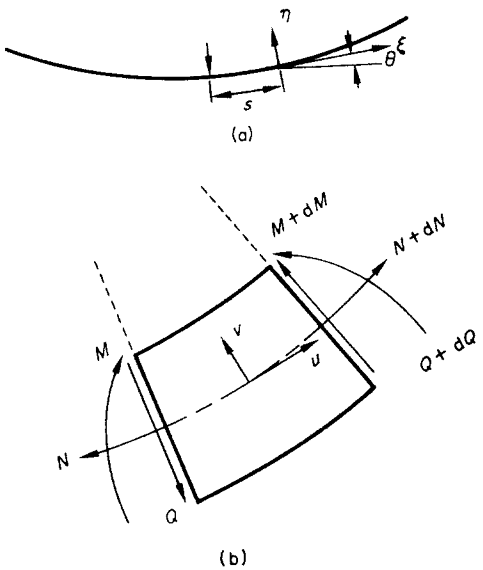

Fig. 1. Beam geometry and stress resultants.

in which E is Young's modulus and $\kappa = k'AG$ where G is the shear modulus and k' is a "shear coefficient" ($k' = 5/6$ and $9/10$ for rectangular and circular beams, respectively). The rates of deformation appearing on the right side of (7a) and (7b) are self-explanatory. The term in parenthesis on the right side of (7c) is the rate of stretching ($\dot{\lambda}/\lambda$) where $\lambda = \mathrm{d}s/\mathrm{d}S$ is the stretch of the centroidal axis. Equation (7c) is equivalent to the relation

$$N = AE\log_e(\lambda). \tag{7c}$$

Equations (5–7) can be put in non-dimensional form by introducing the dimensionless variables:

$$\bar{s} = \frac{s}{L},\ \bar{t} = \frac{t}{(L/c_0)},\ R = c_2/c_0$$

$$\bar{u} = \frac{u}{c_0},\ \bar{v} = \frac{v}{c_2},\ \bar{\omega} = \frac{\omega}{(c_0/L)}$$

$$\bar{N} = \frac{N}{\rho A {c_0}^2},\ \bar{Q} = \frac{Q}{\rho A {c_2}^2},\ \bar{M} = \frac{ML}{\rho I {c_0}^2}$$

where $c_0 = \sqrt{E/\rho}$ is the longitudinal bar velocity and $c_2 = \sqrt{k'G/\rho}$ is the velocity of shear waves in the Timoshenko beam theory; L is a characteristic length for the problem being considered. In terms of dimensionless variables, equations (5–7) become

$$\frac{\partial \bar{M}}{\partial \bar{s}} + \left(\frac{AL^2}{I}\right)R^2\bar{Q} = \frac{\partial \bar{\omega}}{\partial \bar{t}} + \bar{u}\,\frac{\partial \bar{\omega}}{\partial \bar{s}} \tag{8a}$$

$$R^2\,\frac{\partial \bar{Q}}{\partial \bar{s}} + \bar{N}\,\frac{\partial \theta}{\partial \bar{s}} = R\left(\frac{\partial \bar{v}}{\partial \bar{t}} + 2\bar{u}\,\frac{\partial \bar{v}}{\partial \bar{s}}\right) + \bar{u}^2\,\frac{\partial \theta}{\partial \bar{s}} \tag{8b}$$

$$\frac{\partial \bar{N}}{\partial \bar{s}} - R^2 \bar{Q} \frac{\partial \theta}{\partial \bar{s}} = \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial s} - R^2 \bar{v} \frac{\partial \bar{v}}{\partial \bar{s}} - R \bar{v} \bar{u} \frac{\partial \theta}{\partial \bar{s}} \tag{8c}$$

$$\frac{\partial \bar{M}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{M}}{\partial \bar{s}} = \frac{\partial \bar{\omega}}{\partial \bar{s}} \tag{8d}$$

$$\frac{\partial \bar{Q}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{Q}}{\partial \bar{s}} = R \frac{\partial \bar{v}}{\partial \bar{s}} - \bar{\omega} + \bar{u} \frac{\partial \theta}{\partial \bar{s}} \tag{8e}$$

$$\frac{\partial \bar{N}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{N}}{\partial \bar{s}} = \frac{\partial \bar{u}}{\partial \bar{s}} - R \bar{v} \frac{\partial \theta}{\partial \bar{s}} \tag{8f}$$

$$\frac{\partial \theta}{\partial \bar{t}} = R \frac{\partial \bar{v}}{\partial \bar{s}}. \tag{8g}$$

Equations (8) constitute a system of seven first order hyperbolic partial differential equations in seven unknowns. The characteristic speeds $\mathrm{d}\bar{s}/\mathrm{d}\bar{t} = \bar{c}$ are

$$\bar{c} = \bar{u} \pm 1 \text{ (double root)} \tag{9a}$$

$$\bar{c} = \bar{u} \pm \sqrt{R^2 + \bar{N}} \tag{9b}$$

$$\bar{c} = \bar{u}. \tag{9c}$$

These speeds are equivalent to those obtained by Lee [8], provided that proper consideration is given to the minor differences in the two theories, as described in the Introduction.

We seek a solution of equations (8) for the case of a rapidly imposed transverse velocity at an arbitrary section, $\bar{s} = 0$, of an infinite beam. We assume that the beam is initially unstressed and at rest in a horizontal position. From symmetry we need to consider only the semi-infinite region $\bar{s} \geq 0$.

The boundary conditions at $\bar{s} = 0$ are

$$\bar{v}(0, \bar{t}) = \frac{V_0(\bar{t})}{c_2} \tag{10a}$$

$$\bar{u}(0, \bar{t}) = 0 \tag{10b}$$

$$\bar{\omega}(0, \bar{t}) = 0 \tag{10c}$$

where $V_0(\bar{t})$ is the imposed transverse velocity. Conditions (10b) and (10c) result from symmetry. Thus, in the region $\bar{s} \geqslant 0$, $\bar{t} \geqslant 0$, we seek a solution of (8) which vanishes identically for $\bar{t} = 0$ and satisfies the boundary conditions (10) on $\bar{s} = 0$.

Before proceeding with the solution of the stated problem, we note that equations (8) depend on two-dimensional parameters: (AL^2/I) and R. The former parameter can be made to be unity by defining the characteristic length L to be the radius of gyration of the cross-section. The second parameter, $R = c_2/c_0$, is given by

$$R = \sqrt{\frac{k'G}{E}} = \sqrt{\frac{k'}{2(1+\nu)}} \tag{11}$$

where ν is Poisson's ratio. For most materials Poisson's ratio is approximately 0·3. Also, the shear coefficient k' does not change greatly for compact sections. Thus, R does not vary appreciably for compact beams; for the numerical examples we take $R = 0{\cdot}566$ which corresponds to $\nu = 0{\cdot}3$ and $k' = 5/6$ (i.e. a rectangular cross-section). For fixed R, and step function time dependence of the prescribed velocity $V_0(\bar{t})$, the dimensionless solution depends on a single dimensionless parameter v_0/c_2 where v_0 is the imposed velocity.

Numerical solutions of (8) can be obtained by finite difference techniques for various values of v_0/c_2. Small values of v_0/c_2 lead to solutions which, at early times, agree with results from linear Timoshenko beam theory. As v_0/c_2 increases, and the duration of loading increases, the membrane force $\bar{N}$ and its influence on the dynamics of the beam becomes increasingly important. Thus, by comparing solutions for various values of v_0/c_2 it is possible to determine the range of imposed velocity and loading duration for which the linear theory provides an adequate description.

Instead of solving equations (8) directly by means of finite differences it appears that more insight into the influences of finite deflection and membrane force is obtained if the solution is first expanded in powers of $\varepsilon = v_0/c_2$. Hence, all dependent variables in (8) are expanded in the form

$$\bar{M} = \varepsilon M_1(\bar{s}, \bar{t}) + \varepsilon^2 M_2(\bar{s}, \bar{t}) + \cdots \tag{12}$$

Substituting representations of the form (12) into (8) and equating terms of first order in ε we obtain

$$\frac{\partial M_1}{\partial \bar{s}} + R^2 Q_1 = \frac{\partial \omega_1}{\partial \bar{t}} \tag{13a}$$

$$R^2 \frac{\partial Q_1}{\partial \bar{s}} = R \frac{\partial v_1}{\partial \bar{t}} \tag{13b}$$

$$\frac{\partial N_1}{\partial \bar{s}} = \frac{\partial u_1}{\partial \bar{t}} \tag{13c}$$

$$\frac{\partial M_1}{\partial \bar{t}} = \frac{\partial \omega_1}{\partial \bar{s}} \tag{13d}$$

$$\frac{\partial Q_1}{\partial \bar{t}} = R \frac{\partial v_1}{\partial \bar{s}} - \omega_1 \tag{13e}$$

$$\frac{\partial N_1}{\partial \bar{t}} = \frac{\partial u_1}{\partial \bar{s}} \tag{13f}$$

$$\frac{\partial \theta_1}{\partial \bar{t}} = R \frac{\partial v_1}{\partial \bar{s}}. \tag{13g}$$

Also, the boundary conditions (10) become

$$v_1(0, \bar{t}) = 1, \qquad v_i(0, \bar{t}) = 0 \quad \text{for} \quad i \geqslant 2 \tag{14a}$$

$$\left.\begin{aligned} u_i(0, \bar{t}) &= 0, \quad (14\text{b}) \\ \omega_i(0, \bar{t}) &= 0 \quad (14\text{c}) \end{aligned}\right\} \text{for} \quad i \geqslant 1.$$

In order for N_1 and u_1 to satisfy (13c), (13f) and the boundary condition (14b) these functions must be identically zero. The equations governing M_1, Q_1, ω_1 and v_1 are the same as the governing equations in the Timoshenko beam theory employed by Ranganath [6]. The solution of these equations which satisfies the boundary conditions (14a) and (14c) has been obtained by Ranganath [6] by means of Laplace transforms. From (13e) and (13g) the rotation θ_1 can be expressed in terms of Q_1 and ω_1 by

$$\theta_1 = Q_1 + \int_0^t \omega_1 \, \mathrm{d}t. \tag{15}$$

Thus, the solution for the first order terms in expansions of the form (12) can be regarded as known and equal to the linear Timoshenko beam theory solution.

In order to obtain an estimate for the membrane force $\dot{N}$ it is necessary to retain higher order terms in ε in (8c) and (8f). Equating terms of second order in ε we obtain

$$\frac{\partial N_2}{\partial \bar{s}} - R^2 Q_1 \frac{\partial \theta_1}{\partial \bar{s}} = \frac{\partial u_2}{\partial \bar{t}} - R^2 v_1 \frac{\partial v_1}{\partial \bar{s}} \tag{16a}$$

$$\frac{\partial N_2}{\partial \bar{t}} = \frac{\partial u_2}{\partial \bar{s}} - R v_1 \frac{\partial \theta_1}{\partial \bar{s}} \tag{16b}$$

as the governing equations for N_2 and u_2. In principle, an exact solution of equations (16) could be obtained by introducing Laplace transforms, making use of the known transform solutions for Q_1, θ_1, ω_1, and inverting the resulting Laplace transform of the solution N_2, u_2. However, previous experience [6] with evaluation of integrals which arise in the inversion of transform solutions for problems in Timoshenko beam theory indicates that except at stations near the impact face (e.g. $\bar{s} < 50$) the integrand oscillates rapidly and accurate numerical solutions are difficult to obtain. For this reason, it has been found that the second order accurate difference method introduced by Ranganath and Clifton [10] is a more efficient means for obtaining numerical solutions for M_1, Q_1, ω_1, v_1 in the problem defined previously. Thus, it appears that the difference method [10] should be more efficient than the Laplace transform method for obtaining numerical solutions for N_2 and u_2. Consequently, numerical solutions of (16) as well as (13), are obtained by means of the finite difference method [10] which makes use of integration of ordinary differential equations along characteristics. The incremental relations along characteristics corresponding to equations (13a), (13b), (16a), (13d), (13e), (16b), and (13g) can be shown, using well known procedures (e.g. Courant and Hilbert [11]), to be

$$\mathrm{d}M_1 \mp \mathrm{d}\omega_1 \pm R^2 Q_1 \mathrm{d}\bar{t} = 0 \quad \text{along} \quad \frac{\mathrm{d}\bar{s}}{\mathrm{d}\bar{t}} = \pm 1 \tag{17a}$$

$$\mathrm{d}Q_1 \mp \mathrm{d}v_1 + \omega_1 \, \mathrm{d}\bar{t} = 0 \quad \text{along} \quad \frac{\mathrm{d}\bar{s}}{\mathrm{d}\bar{t}} = \pm R \tag{17b}$$

$$\mathrm{d}\theta_1 - \mathrm{d}Q_1 - \omega_1 \, \mathrm{d}\bar{t} = 0 \quad \text{along} \quad \frac{\mathrm{d}\bar{s}}{\mathrm{d}\bar{t}} = 0 \tag{17c}$$

$$\mathrm{d}N_2 \mp \mathrm{d}u_2 \pm \frac{R^3 Q_1}{(1-R^2)} \mathrm{d}v_1 - \frac{R^4 Q_1}{(1-R^2)} \mathrm{d}Q_1 + (\pm R v_1 - R^2 Q_1)\, \mathrm{d}\theta_1 - \frac{R^4 Q_1}{(1-R^2)} \omega_1 \, \mathrm{d}\bar{t} = 0 \quad \text{along} \quad \frac{\mathrm{d}\bar{s}}{\mathrm{d}\bar{t}} = \pm 1. \tag{17d}$$

Equations (17a–17c) determine M, Q_1, ω_1, v_1, and θ_1 whereas equation (17d), determine N_2 and u_2 when solved simultaneously with (17a–17c).

Numerical solutions of (17) have been obtained by means of the difference method [10] which is an extension to second order accuracy of the method introduced by Courant, Isaacson and Rees [12]. Since the method applies directly only to problems in which the solution is continuous, the jump in the imposed transverse velocity at $\bar{t} = 0$ was eliminated by introducing a finite rise time for the imposed velocity to attain the value v_0. The function $V_0(\bar{t})$ was assumed to have the parabolic form

$$V_0(\bar{t}) = v_0\left(2\frac{\bar{t}}{\bar{t}_0} - \frac{\bar{t}^2}{\bar{t}_0{}^2}\right) \tag{18}$$

where $\bar{t}_0$ is a dimensionless risetime. The value $\bar{t}_0 = 40$ was assumed for the numerical computations; the solution at times later than $\bar{t}_0$ after the arrival of the wave front is relatively insensitive to the assumed value for $\bar{t}_0$. The nondimensional mesh sizes $\Delta\bar{s}$ and $\Delta\bar{t}$ were both taken to be equal to 2·5. Numerical trials with larger mesh sizes indicate that larger mesh sizes could be used without significant loss in accuracy.

DISCUSSION OF RESULTS

The functions determined by the integration of equations (17) do not depend on the small parameter $\varepsilon = v_0/c_2$. Therefore, an approximate solution to the original problem is obtained for a range of impact velocities by solving equations (17) only once and then making use of expansions of the form (12). In this approximate solution the quantities $\overline{M}$, $\overline{Q}$, $\bar{\omega}$, $\bar{v}$, $\bar{\theta}$ are proportional to ε whereas $\overline{N}$ and $\bar{u}$ are proportional to ε^2. Thus, for example, as ε increases the stresses due to the normal force $\overline{N}$ become more important relative to the stresses due to the bending moment $\overline{M}$. A direct comparison of the relative magnitudes of the stresses due to these two generalized forces is the ratio $\overline{N}/\overline{M}$ which, from the definitions of the dimensionless quantities, is equal to the ratio of the longitudinal stress due to the normal force N to the longitudinal stress (at a distance L from the centroidal axis) due to the bending moment M. The ratio $\overline{N}/\overline{Q}$ is $(1/R^2)$ times the ratio of the longitudinal stress due to N to the average shear stress on the cross-section.

Figures 2 and 3 show the time dependence of the approximate generalized stresses $\overline{M} \simeq \varepsilon M_1$, $\overline{Q} \simeq \varepsilon Q_1$, $\overline{N} \simeq \varepsilon^2 N_2$ at two stations and for three impact velocities. From Fig. 2, the impact velocity $v_0/c_2 = 0{\cdot}015$ corresponds to an extreme fiber stress in bending at $\bar{s} = 0$ which is approximately 1·5% of the elastic modulus E for the material (i.e. stress $\sigma = Md/(2I) = \sqrt{3}\,\overline{M}E \simeq 0{\cdot}015E$). Since this stress is beyond the yield stress in essentially all metals, larger impact velocities would require a theory which includes plastic deformation (see e.g. [13]). Because $\overline{N}$ in Fig. 2 is a monotonic increasing function of time whereas $\overline{M}$ becomes constant after an initial risetime, the solution predicts that the relative importance of $\overline{N}$ increases with increasing time. At $v_0/c_2 = 0{\cdot}015$ the value of $\overline{N}$ at the latest time shown in Fig. 2 corresponds to a normal stress of $0{\cdot}000517E$, which is approximately 3·5% of the maximum stress $0{\cdot}015E$ due to bending. Thus, even at the highest impact velocity and the latest time considered, the stress due to the normal forces is small relative to that due to bending. At sufficiently late times the stresses due to the normal force finally dominate those due to bending. However, the duration for which the perturbation solution remains valid has not been established.

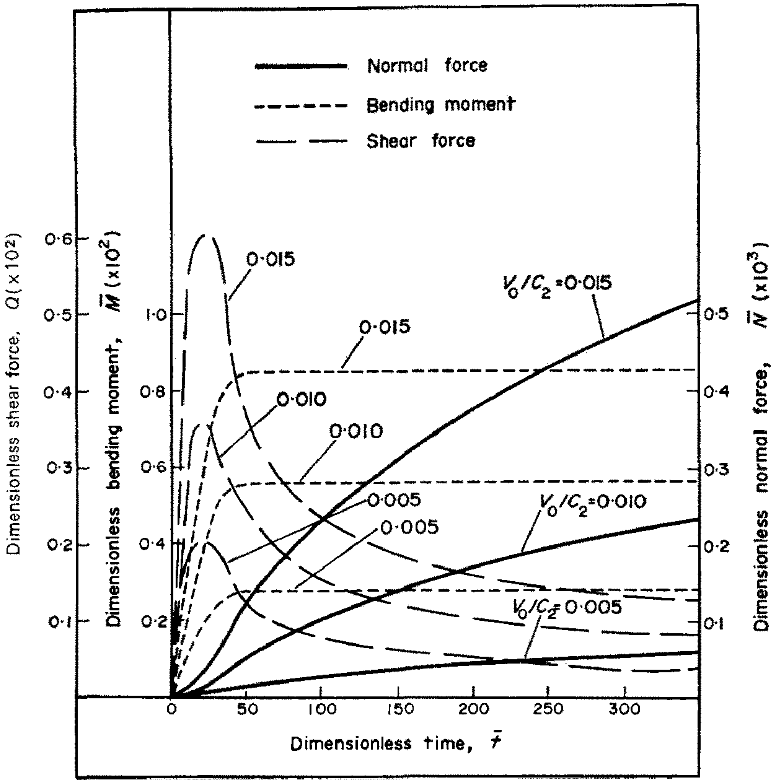

Fig. 2. Stress–time profiles at $\bar{s} = 0$.

The oscillatory character of the time dependence of the bending moment shown in Fig. 3 agrees with experimental results shown in Fig. 3 of [6]. The dimensionless distance $\bar{s} = 50$ corresponds to a distance of 1·80 in. for the 1/8 in. thick beams used in [6] and the shape of the moment time profiles lies between the shapes observed in the experiments at distances of 1·50 and 2·00 in. The calculations show that, as expected from characteristics theory, the wavefront of bending moment and shear actually propagates at the characteristic speed $d\bar{s}/d\bar{t} = 1$; however, the amplitude near the wavefront is negligibly small so that the first detectable part of the bending and shear disturbance propagates with a velocity slightly less than 0·5 c_0. This aspect of the solution is also in agreement with the strain–time profiles shown in Fig. 3 of [6]. The time dependence of the normal force at $\bar{s} = 50$ is nearly the same as at $\bar{s} = 0$ and is indicative of the weak dispersion exhibited by the computed longitudinal wave. The amplitude of the impact velocity in [6] is slightly less than the lowest value for which stress–time profiles are given in Fig. 3. Therefore, the predicted strain associated with extension due to the normal force N is less than 6×10^{-5} (i.e. strain $\varepsilon = (N/A)/E = \bar{N} \leqslant 6 \times 10^{-5}$) which is less than 0·5% of the extreme fiber strain (i.e. $\sqrt{3}\,\bar{M}$) of $1{\cdot}6 \times 10^{-2}$ which occurs at the first positive peak in Fig. 3 for $v_0/c_2 = 0{\cdot}005$. The small strains associated with the normal force correspond to a vertical displacement of less than a line width on the scale of the experimental strain–time profiles in Fig. 3 of [6]. Thus, it is not surprising that such overall extensional strains were not observed even when experiments were carried out specifically to investigate the significance of string forces [14].

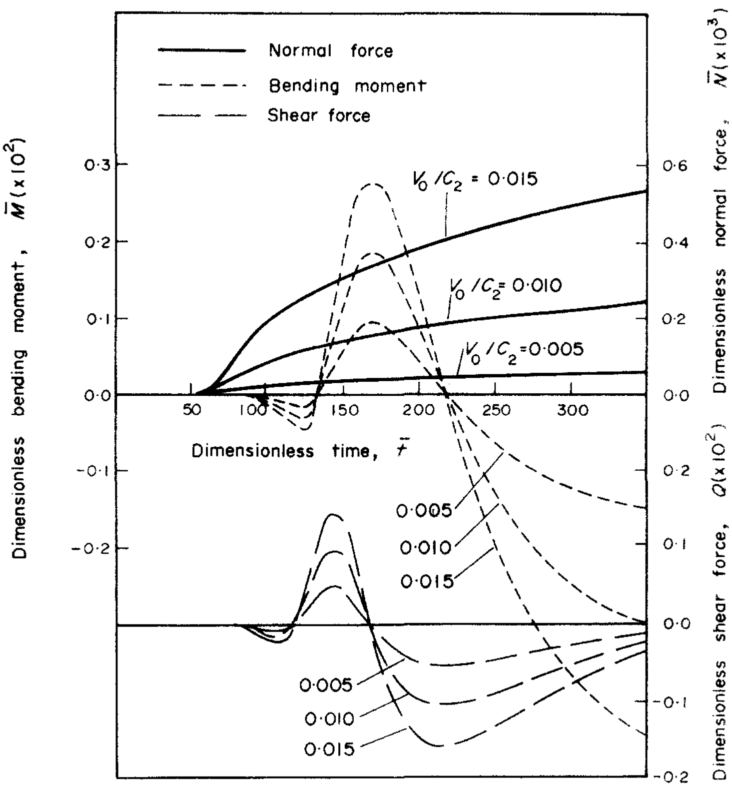

Fig. 3. Stress–time profiles at $\bar{s} = 50$.

The dependence of the generalized stresses on distance at a fixed time is shown in Fig. 4. The moment and shear distributions are oscillatory and essentially 90° out of phase. The normal force varies slowly with distance, except near the wavefront $\bar{s} = \bar{t}$. The longitudinal particle velocity u_2 is negative for $0 < \bar{s} < \bar{t}$; the spatial dependence of u_2 is similar to that of N_2 except that u_2 goes sharply to zero near $\bar{s} = 0$ in order to satisfy the boundary condition (14b). In the region in front of the bending and shear disturbances in Fig. 4, the propagation of the longitudinal wave is essentially non-dispersive with

$$N_2(\bar{s}, \bar{t}) + u_2(\bar{s}, \bar{t}) \simeq 0 \tag{19a}$$

$$N_2(\bar{s}, \bar{t}) - u_2(\bar{s}, \bar{t}) \simeq N_2(\bar{s}_b, \bar{t} - (\bar{s} - \bar{s}_b)) - u_2(\bar{s}_b, \bar{t} - (\bar{s} - \bar{s}_b)) \tag{19b}$$

where $\bar{s}_b$ is a position near the front of the detectable bending disturbance. Equations (19) follow directly from equations (17d) in regions where Q_1, v_1, θ_1, ω_1 are negligibly small.

Equations (19) can be used to give a physical interpretation of the normal force distribution (e.g. Fig. 4) which arises under transverse impact. Between the point of impact and the front of the bending wave the centroidal axis is extended due to transverse displacement. This extension produces a tensile force which is propagated forward at the longitudinal wave speed $d\bar{s}/d\bar{t} = 1$. Beyond the front of the bending wave the longitudinal wave becomes essentially non-dispersive, in agreement with the elementary one-dimensional theory of

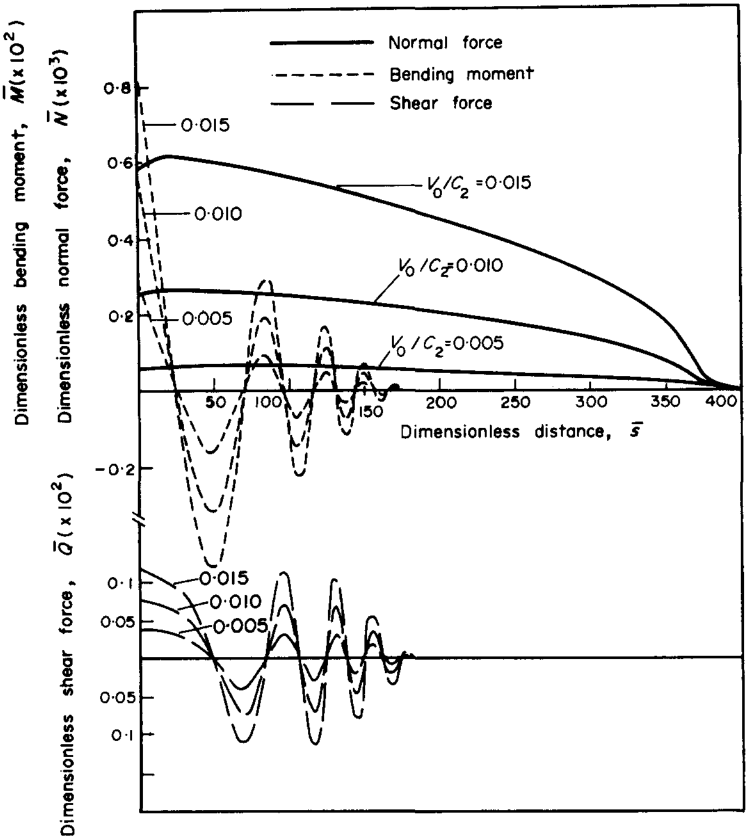

Fig. 4. Stress–distance profiles at $\bar{t} = 400$.

longitudinal waves in bars. Thus, the longitudinal wave is generated in the region where bending is occurring, but propagates on in front of the bending wave because the speed of the longitudinal wave is greater than that of the dispersive bending wave.

CONCLUSIONS

By an appropriate choice of non-dimensional variables and a consistent approximation neglecting higher order terms, the dynamic deflection of an initially straight beam subjected to transverse impact is shown to depend on a combined geometrical and material parameter which does not vary significantly and a loading parameter which determines the amplitude of the motion. The normal or string force is proportional to the square of the impact velocity and at any station increases monotonically with increasing time. Within the limits of the approximate theory presented here, the elementary Timoshenko theory predicts the bending moment and the shear force adequately even when there is significant stretching of the centroidal axis. For impact velocities which are sufficiently small for the assumption of linear elastic material behavior to be applicable, the stresses associated with the string force are negligible relative to the maximum bending stresses provided that the loading duration is not greater than approximately 10^3 times the transit time of waves through the beam thickness.

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Абстракт — Решается вопрос бесконечной эластичной балки, подвергаемой в одной точке граничным условиям постоянной скорости. Включаются эффекты: финитных отклонений, реакции мембраны, вращательной инерции и деформации сдвига. Уравнения проблемы переводятся в безразмерную форму и для получения подходящего приближения применяется аппроксимация возмущения. Численные решения при различных скоростях контакта получили для изгибающего момента, для силы сдвига и для реакции мембраны. Нашли, что решение проблемы зависит от совокупности геометрического и материального параметров, которые для компактных участков существенно не различаются и от параметра нагрузки, который определяет амплитуду реакции. И в конечном счете, теория линейной балки Тимошенко оказывается предсказывает изгибающий момент и силу сдвига удивительно точно, даже, если отклонения достаточно сильные, чтобы причинить заметное растяжение центроидной оси.